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GRAPH EQUIVARIANT COHOMOLOICAL RIGIDITY FOR GKM-GRAPHS

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ABSTRACT. This is the report of the author's talk delivered at the RIMS conference "Geometry, Algebra and Combinatorics in Transformation group theory". The main theorem of [10] and the philosophy governing *loc.cit* are explained.

1. EQUIVARIANT COHOMOLOGY, AS INVARIANT

In the book A. Borel introduced the notion of the equivariant cohomology $H_G^*(X)$ of a G-space X. In *loc.cit* Borel also examined the usage of the cohomology theory H_G^* to the Smith conjecture.

Since 1980, equivariant cohomology has been applied in different research areas. In the seminal work [1], Atiyah-Bott studied torus equivariant cohomology in its relation to moment map. Kazhdan-Lusztig [7] found a construction of the Springer representation via equivariant cohomology. Nowadays equivariant cohomology is frequently used in mathematical physics.

Among these big applications, what we want to discuss here is yet another aspect of torus equivariant cohomology, that is, as invariant.

Wheres many applications were found after the work of Borel, any research in this direction (seemingly) has been missed before the last century.

At least for my knowledge, the following result of Masuda [9] is the first result in this direction:

Theorem 1.1 (Masuda)**.** Let X and Y be n-dimensional toric manifolds (i.e., nonsingular complete (normal) toric variaties over $\mathbb{C}$). Then the following two conditions are equivalent:

(i) $X \cong Y$ as $(\mathbb{C}^\times)^n$-varieties.
(ii) $H^*_{(\mathbb{C}^\times)^n}(X) \cong H^*_{(\mathbb{C}^\times)^n}(Y)$ as graded $H^*(B(\mathbb{C}^\times)^n)$-algebras.

Remark 1.2. (i) As a corollary, one finds that if two toric manifolds have the same $(\mathbb{C}^\times)^n$-equivariant homotopy type, then they are isomorphic as $(\mathbb{C}^\times)^n$-*varieties*. This well-explains the power of torus equivariant cohomology as invariant.
(ii) Since toric manifolds are in one-to-one correspondence to non-singular complete fans, Theorem 1.1 can be viewed as equivariant cohomological rigidity of such fans.
(iii) Toric hyperKähler analogue also holds. See [6].

While Theorem 1.1 seems a speciality of toric geometry, I believe that this kind of equivariant rigidity holds for much wider class of $(S^1)^r$-manifolds.

Our central dogma is the following. Let T be a compact torus.

Working hypothesis

A T-manifold and the graded $H^*(BT)$-algebra $H_T^*(X)$ are of equal value.

Our problems are two hold:

(i) To find a wide class of T-manifolds in which equvariant cohomological rigidity holds.
(ii) To reveal why such rigidity phenomenon happens.

The content of this report is concerning the former question.

Theorem 1.1 tells us that the working hypothesis is true for toric manifolds. Our main result (= Theorem 2.5 below) extends Theorem 1.1 to general abstract GKM-graphs, a generalization of non-singular complete fans.

2. Abstract GKM-graph

The class of *GKM-manifolds* is a vast generalization of that of toric manifolds. Historically it was introduced by Goresky-Kottwitz-MacPherson [4] in algebro-geometric setting. Later Guillemin-Zara [5] reformulated it in the category of closed $(S^1)^r$-manifolds. Moreover, Guillemin-Zara found a framework which allows us possible to study torus equivariant cohomology of equivariantly formal GKM-manifolds in purely combinatorial fashion.

This section is devoted to recalling their formulation. Let $\mathcal{G}$ be a finite n-valent graph (multi-edges are allowed, but loops are not) and $\mathcal{V}$ be the set of vertexes. We denote by $\mathcal{E}$ the set of *directed* edges (thus the cardinality of $\mathcal{E}$ is even). Let $\mathcal{G}$ be a finite n-valent undirected graph (multi-edges are allowed, but loops are not) with vertex set $\mathcal{V}$. We denote by $\mathcal{E}$ the set of directed edges of $\mathcal{G}$. (Note that $\mathcal{E}$ is not the set of edges of $\mathcal{G}$; the cardinality of $\mathcal{E}$ is twice that of the edge set.) For each $e \in \mathcal{E}$, we denote by $\overline{e}$ the directed edge obtained by reversing the direction of e. Let $i(e)$ and $t(e)$ be the initial and terminal point of a directed edge e, respectively. We set

$$\mathcal{E}_p := \{\, e \in \mathcal{E} \mid i(e) = p \,\}$$

for any vertex p.

Definition 2.1. A map $\alpha\colon \mathcal{E} \to H^2(BT)$ is called an *axial function* on $\mathcal{G}$ if it satisfies the following three conditions for all e, $e' \in \mathcal{E}$:

(i) $\alpha(\overline{e}) = \pm\alpha(e)$.
(ii) (*GKM condition*) $\alpha(e)$ and $\alpha(e')$ are linearly independent over $\mathbb{Z}$ if $e \neq e'$ and $i(e) = i(e')$.
(iii) (*Primitivity*) The greatest common divisor of the coefficients of $\alpha(e)$ is 1.

The following notion, found by Gullemin-Zara [5], is a corner stone in GKM-theory:

Definition 2.2. Let α be an axial function on $\mathcal{G}$. A *parallel transport* of $(\mathcal{G}, \alpha)$ is a family $\mathcal{P} = \{\mathcal{P}_e\}_{e\in\mathcal{E}}$ of bijections $\mathcal{P}_e\colon \mathcal{E}_{i(e)} \to \mathcal{E}_{t(e)}$ satisfying the following conditions for all $e \in \mathcal{E}$ and all $e' \in \mathcal{E}_{i(e)}$:

(i) $\mathcal{P}_{\overline{e}} = \mathcal{P}_e^{-1}$.
(ii) $\mathcal{P}_e(e) = \overline{e}$.
(iii) $\alpha(\mathcal{P}_e(e')) - \alpha(e') \in \mathbb{Z}\,\alpha(e)$.

Definition 2.3. (1) A pair $(\mathcal{G}, \alpha)$ is called an *abstract GKM-graph* if it admits at least one connection.

(2) The *graph equivariant cohomology* $H_T^*(\mathcal{G})$ is defined to be

$$\left\{ f : \mathcal{V} \to \mathbb{Z}[x_1, \cdots, x_r] \,\middle|\, f(i(e)) - f(t(e)) \text{ is divisible by } \alpha(e) \text{ for all } e \in \mathcal{E} \right\}.$$

Remark 2.4. (i) For a GKM-manifold X, one can associate an abstract GKM-graph $\mathcal{G}_X$ by encording the graph structure of the 1-skeleton of X and the weights of tangencial representations. $\mathcal{G}_X$ is called the GKM-graph of X. If X satisfies the so-called equivariant formality, the torus equivariant cohomology $H_T^*(X)$ is isomorphic to $H_T^*(\mathcal{G}_X)$ as graded $H^*(BT)$-algebras (see [5] for details). This is why $H_T^*(\mathcal{G})$ is called graph "equivariant cohomology".

(ii) Any toric manifold is an equivariantly formal GKM-manifold. In addition, its GKM-graph is essentially the same to the corresponding fan. In this case, its graph equivariant cohomology is known to be isomorphic to the Stanley-Reisner ring as graded rings.

(iii) Above notation and terminology are somewhat different from usual one. See [10], Remark 2.4.

Our main theorem is the following:

Theorem 2.5. Let $\mathcal{G}, \mathcal{G}'$ be abstract GKM-graphs with the same type.

(i) $\mathcal{G} \cong \mathcal{G}$ as GKM-graphs, i.e., these are isomorphic as graphs and the corresponding edges have the same weight up to multiplication by ± 1.

(ii) $H_T^*(\mathcal{G}) \cong H_T^*(\mathcal{G}')$ as graded $\mathbb{Z}[x_1, \ldots, x_r]$-algebras.

Remark 2.6. Masuda's theorem (=Theorem 1.1) opened the door to the so-called *cohomological rigidity problem* in toric topology. In view of Theorem 2.5, it is seemingly valuable to study a similar problem for equivariantly formal GKM-*varieties*.

3. On the proof of Theorem 2.5

In this section we explain the idea for proving Theorem 2.5.

The difficultly is that it is highly difficult (and seemingly impossible) to express graph equivariant cohomology $H_T^*(\mathcal{G})$ in terms of generators and relations, in a uniform way.

To overcome this point, we recall our working hypothesis, that is,

$(*)$ T-space X and $H^*(BT)$-algebra $H_T^*(X)$ are of equal value.

Once we believe $(*)$, there should exist the notion of "1-skeleton of $H_T^*(X)$". Since the 1-skeleton of X takes a very small part of X, "the 1-skelton of $H_T(X)$" may be a very small subset of $H_T^*(X)$ similarly. This expectation leads us the following definition:

Definition 3.1. Let p, q be distinct vertexes of a GKM-graph $\mathcal{G}$. We define the *1-ideal* I_{pq} associated with p, q by

$$I_{pq} := \{f \in H_T^*(\mathcal{G}) | f(r) = 0 \ (r \in \mathcal{V} \setminus \{p, q\})\}.$$

We regard the set of 1-ideals I_{pq} for adjacent vertexes p, q as "the 1-skelton of $H_T^*(X)$".

Next, assume that an algebra isomorphism $H_T^*(\mathcal{G}) \to H_T^*(\mathcal{G}')$ is given. Under $(*)$ the algebra isomorphism corresponds to a T-equivariant homeomorphism $X' \to X$. The T-equivariant homeomorphism induces a T-equivariant homeomorphism $X_1' \to$

X_1. Back to the algebraic side, the existence of the T-equivariant homeomorphism $X_1' \to X_1$ indicates that 1-ideals should be related under the algebra isomorphism:

Lemma 3.2. For any algebra isomorphism $\varphi : H_T^*(\mathcal{G}) \to H_T^*(\mathcal{G}')$ and any 1-ideal I of $H_T^*(\mathcal{G})$, the image $\varphi(I)$ is a 1-ideal of $H_T^*(\mathcal{G}')$

Above Lemma implies condition (i) in Theorem 2.5. See [10] for details.

Note that everything goes by logically if we accept the working hypothesis $(*)$.

Remark 3.3. Recently an alternative proof of Theorem 2.5 is found by Matthias Franz. The detail will appear as a joint note [3].

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